

Deconstructing holographic liquids

Dominik Nickel and Dam T. Son

*Institute for Nuclear Theory, University of Washington,
Seattle, Washington 98195-1550, USA*

(Dated: September 2010)

We argue that there exist simple effective field theories describing the long-distance dynamics of holographic liquids. The degrees of freedom responsible for the transport of charge and energy-momentum are Goldstone modes. These modes are coupled to a strongly coupled infrared sector through emergent gauge and gravitational fields. The IR degrees of freedom are described holographically by the near-horizon part of the metric, while the Goldstone bosons are described by a field-theoretical Lagrangian. In the cases where the holographic dual involves a black hole, this picture allows for a direct connection between the holographic prescription where currents live on the boundary, and the membrane paradigm where currents live on the horizon. The zero-temperature sound mode in the D3-D7 system is also re-analyzed and re-interpreted within this formalism.

I. INTRODUCTION

There is considerable interest in using holographic methods [1–3] to study strongly coupled quantum liquids. A typical example of such a liquid is the $\mathcal{N} = 4$ super-Yang-Mills plasma, which is frequently used as a prototype for the strongly coupled quark gluon plasma created at RHIC [4]. The liquids are described mathematically as a solution of a higher-dimensional theory in an asymptotically AdS spacetime. To compute correlations functions using gauge/gravity duality, one solves field equations in the bulk with boundary conditions at the AdS boundary. The microscopic theory is typically a large- N gauge theory in the strong coupling limit.

Frequently, however, one is not interested in the details of the microscopic theory, but only in the long-distance behavior at finite temperature and/or density. This is the regime relevant for the hydrodynamic behavior at finite temperature and quantum critical behaviors at zero temperature. In fact, much of the recent “AdS/CMT” activities [5, 6] are directed toward finding new quantum critical behaviors. One can then ask: what are the minimal ingredients needed to describe the long-distance dynamics of holographic liquids? Is the full holographic description needed? As we will argue in this paper, it is not. The long-distance behavior of holographic liquids can be described by a set of Goldstone bosons, interacting with a strongly coupled infrared sector. Holography may be needed to describe the infrared sector, but not the Goldstone bosons.

That a strongly coupled infrared sector should appear in the low-energy effective theory is rather clear. According to the dictionary of holography, low energies correspond to the

near-horizon part of the metric. Various possible types of behavior of the metric in the IR have been observed and classified [7–9], and one expects different IR asymptotics to correspond to different IR sectors. For example, a black hole event horizon corresponds to a thermal bath, and AdS asymptotics to a conformal field theory. The AdS₂ infrared asymptotics of the Reissner-Nordström metric, which is supposed to describe a finite-density, zero-temperature system, should correspond to a (0+1)-dimensional conformal field theory, although the nature of such a theory is not very clear. It has been seen explicitly in many calculations that the near-horizon geometry influences the singular behavior of the inverse propagators [10].

Nevertheless, the calculation of the full propagator always involves the whole metric, not just its near-horizon part [10–12]. One can argue, rather generally, that the near-horizon geometry cannot contain complete information about the long-distance physics. Consider, for example, an extremal Reissner-Nordström black hole, holographically dual to a finite-density medium. This medium is compressible, as seen from its equation of state, and should support a gapless (for example, propagating or diffusive) mode related to charge transport (these modes are seen explicitly in two-point Green functions [11, 12]). However, the AdS₂ metric cannot support such a mode, as the spatial coordinates factor out of it. Another case is a holographic liquid where the infrared metric is an AdS metric, but with a different speed of light [7]. In such a liquid the conformal Ward identity $T^\mu_\mu = 0$ (with the vacuum speed of light) should remain valid in the long-distance regime. But the near-horizon metric in this case has a different speed of light than the one appearing in the Ward identity, and it is not clear how low-energy physics “knows” about the real light speed.

In this paper, we suggest that the long-distance description of holographic liquids involve a set of Goldstone bosons in addition to the degrees of freedom living in the near horizon region. We can visualize the process of finding the low-energy effective theory as a Wilsonian renormalization group procedure. In this language, the Goldstone boson appears as the only mode living outside the near-horizon part of the metric that survives this procedure. In the simplest case of particle number diffusion, the Goldstone boson arises from the spontaneous breaking of a $U(1) \times U(1)$ symmetry down to the diagonal $U(1)$. One of the $U(1)$ is that of a conserved charge, but the other $U(1)$ is an emergent dynamical $U(1)$ gauge field. The holographic infrared degrees of freedom, living in the near horizon part of the metric (which, for shortness, will be called just the IR degrees of freedom) are coupled to the dynamic $U(1)$ field, but are not coupled directly to the particle number $U(1)$ field. This is summarized in the “moose diagram” of Fig. 1.

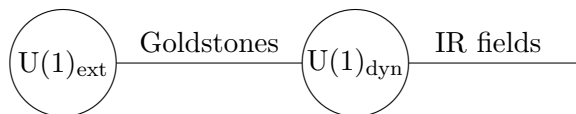


FIG. 1: The moose diagram for holographic liquids.

Our picture is similar to the “semiholographic” models considered in Ref. [13]. The

difference is that Ref. [13] concerns mostly with probe fermion fields, but we are interested in the degrees of freedom transporting charge and energy-momentum. The emphasis on the Goldstone modes distinguishes this paper from other works seeking to relate the properties of the boundary theory and the horizon [14–16],

The dynamic gauge field connecting the Goldstone boson and the IR fields bears some resemblance to the emergent gauge fields in some condensed-matter models [17]. It suggests that the holographic constructions and condensed matter models involving emergent gauge fields are closer to each other than previously thought. Similar connections have been explored in Ref. [18]. One interesting fact that we found is the appearance of dynamic gravity in the low-energy effective theories arising from holography. Note that there have been attempts to construct lattice models that would give rise to gravity [19].

The paper is organized as follows. In Sec. II we consider the simplest problem: a gauge field in a fixed black-brane metric. We show that the diffusion mode can be interpreted as a Goldstone boson, which is coupled, through an emergent gauge field, to a stretched horizon with a finite electrical conductivity. In Sec. III we tackle a more difficult problem of gravitational fluctuations. We show that the low-energy dynamics is that of a Goldstone boson coupled to an emergent metric. We show how the viscosity of the stretched horizon becomes, through the Goldstone boson, the viscosity at the boundary. A by-product of this Section is a bi-gravity formulation of hydrodynamics. In Sec. IV we give the Goldstone-boson interpretation to the zero-temperature sound (zero sound) found in Ref. [20]. We conclude with Sec. V.

II. DIFFUSION FROM GOLDSTONE BOSON DYNAMICS

We illustrate the picture advocated above on the example of charge diffusion at finite temperature. The gravitational description involves a gauge field A_μ in a black hole horizon,

$$S = -\frac{1}{4g_{\text{YM}}^2} \int d^5x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (1)$$

The metric will be chosen in the form

$$ds^2 = -r^2 f(r) dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2 f(r)}, \quad (2)$$

where $f(r_0) = 0$ at the horizon $r = r_0$, and $f(\infty) = 1$.

We are interested in the dynamics at distances larger than some scale. This regime, following the holographic dictionary, maps onto a region near the black hole horizon (the shaded region in Fig. 2). We then choose an arbitrary r_Λ as a coordinate separating the near-horizon region from the outside region. The action therefore is the sum of two actions,

$$S = S_{\text{IR}} + S_{\text{UV}}. \quad (3)$$

The gauge field $A_\mu = A_\mu(r = \infty)$ at the AdS boundary couples to S_{UV} only. The value of A_μ at r_Λ , $a_\mu = A_\mu(r_\Lambda)$, serves as a source for the IR theory. This source also couples to the

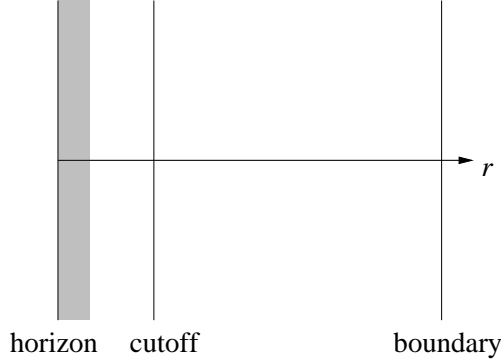


FIG. 2: The division of space into two regions. The holographic IR modes live in the shaded region well below the cutoff.

UV degrees of freedom: the UV theory has two boundaries and is coupled to two external gauge fields

$$S = S_{\text{IR}}[a_\mu, \phi_{\text{IR}}] + S_{\text{UV}}[A_\mu, a_\mu, \phi_{\text{UV}}]. \quad (4)$$

Here ϕ_{UV} and ϕ_{IR} denote fields in the UV and IR theories, respectively. The field a_μ should be determined by the equation of motion, i.e., by the condition that the variation of the total action with respect to it is zero: $\delta S / \delta a_\mu = 0$. In the quantum theory, one should perform a path integral over a_μ .

The key observations are that (i) the UV theory is a confining theory and can be rewritten as a theory of mesons, and (ii) the only meson important at low energies is the Goldstone boson arising from the breaking of $U(1) \times U(1)$ symmetry down to the diagonal $U(1)$ group. This Goldstone boson arises in a manner similar to the pion in the Sakai-Sugimoto model [21]. To find the pions, we note that, if one fixes the values the temporal and spatial components of the gauge field A_μ ($\mu \neq r$) on the two boundaries, then the Wilson line

$$\phi = \int_{r_\Lambda}^{\infty} dr A_r(r, x) \quad (5)$$

is invariant under all gauge transformations preserving the boundary values of A_μ up to a global transformation. The value of the Wilson line is the Goldstone boson field. Alternatively, if one works in the radial gauge $A_r = 0$, one cannot impose the Dirichlet boundary conditions $A_\mu = 0$ ($\mu \neq r$) on both boundaries. If $A_\mu = 0$ on one boundary, then it should be $A_\mu = \partial_\mu \phi$ on the other. The Goldstone boson in the radial gauge is that gauge parameter ϕ .

Given the symmetries, we can write down the action

$$S = \int d^4x \frac{1}{2} [f_t^2 (\partial_0 \phi - A_0 + a_0)^2 - f_s^2 (\partial_i \phi - A_i + a_i)^2] + S_{\text{IR}}[a_\mu]. \quad (6)$$

Here f_t and f_s are some low-energy constants that will be determined later.

The IR theory $S_{\text{IR}}[a_\mu]$ is defined, holographically, as the theory dual to a $U(1)$ field on a black brane horizon. The only information about this theory that we will need is its response

to a_μ , which is an external gauge field from the point of view of the IR degrees of freedom. The relationship is found within the black hole membrane paradigm [22–24], which attaches a finite electrical conductivity σ to the stretched horizon at $r = r_\Lambda$,

$$\mathbf{j}_{\text{IR}} = \frac{\delta S_{\text{IR}}}{\delta \mathbf{a}} = \sigma \mathbf{e}, \quad e_i = -f_{0i} = -\partial_0 a_i + \partial_i a_0. \quad (7)$$

In holography, the conductivity arises from the incoming-wave boundary condition at the horizon [14]. We do not need to specify the charge density $j^0 = \delta S_{\text{IR}}/\delta a_0$; it is determined by the conservation law on the horizon, $\partial_0 j^0 + \partial_i j^i = 0$.

We pause here to clarify one subtlety. The dynamics on the horizon is dissipative, therefore the equation $j = \delta S/\delta a$ is not valid in the strict sense. The precise meaning of this equation is found in the closed-time-path formalism in the RA basis [25], where j is understood as j_R and a as a_A . For the sake of writing down the field equation, our naive equation is sufficient.

The extremization of the action Eq. (6) with respect to a_i gives (we set the external field $A_\mu = 0$):

$$\frac{\delta S}{\delta a_i} = \frac{\delta S_{\text{UV}}}{\delta a_i} + \frac{\delta S_{\text{IR}}}{\delta a_i} \equiv j^i + j_{\text{IR}}^i = -f_s^2(\partial_i \phi + a_i) + \sigma e_i = 0. \quad (8)$$

This is one equation of motion. The other equation of motion is obtained by varying (6) with respect to ϕ ,

$$f_t^2 \partial_0(\partial_0 \phi + a_0) - f_s^2 \partial_i(\partial_i \phi + a_i) = 0. \quad (9)$$

We can write the equations in terms of the currents $j^0 = f_t^2(\partial_0 \phi + a_0)$, $j^i = -f_s^2(\partial_i \phi + a_i)$:

$$\partial_0 j^0 + \partial_i j^i = 0, \quad (10)$$

$$j^i = -\frac{\sigma}{f_s^2} \partial_0 j^i - \frac{\sigma}{f_t^2} \partial_i j^0. \quad (11)$$

In the low-frequency regime ($\omega \ll f_s^2/\sigma$), the first term in the right-hand-side of Eq. (11) is negligible. We then obtain a diffusion equation for j^0 ,

$$\partial_0 j^0 - D \nabla^2 j^0 = 0, \quad (12)$$

where the diffusion constant D is related to the membrane electric conductivity σ and the susceptibility f_t^2 as $D = \sigma/f_t^2$. Note that f_s^2 does not enter this final expression.

Calculating the parameters of the effective theory

To find the value of f_t^2 and f_s^2 , we match the effective field theory with holographic calculations. If we freeze the Goldstone boson to $\phi = 0$ and turn on constant external A_0 and A_i , then the coefficients f_t^2 and f_s^2 are obtained by expanding S to quadratic order in the external fields,

$$S = \frac{1}{2}(f_t^2 A_t^2 - f_s^2 A_i^2). \quad (13)$$

Freezing the Goldstone boson at $\phi = 0$ corresponds to working in the radial gauge $A_r = 0$ and putting $A_\mu = 0$ at the horizon. The equation satisfied by A_t and A_i are then

$$\partial_r(r^3 \partial_r A_t) = 0, \quad \partial_r[r^3 f(r) \partial_r A_i] = 0. \quad (14)$$

Solving the equations and substituting into the action, we then find f_t^2 and f_s^2 ,

$$f_t^2 = \frac{1}{g_{\text{YM}}^2} \left[\int_{r_\Lambda}^{\infty} \frac{dr}{r^3} \right]^{-1}, \quad f_s^2 = \frac{1}{g_{\text{YM}}^2} \left[\int_{r_\Lambda}^{\infty} \frac{dr}{r^3 f(r)} \right]^{-1}. \quad (15)$$

We notice here that f_t^2 remains finite in the limit $r_\Lambda \rightarrow r_0$ but, since $f(r)$ vanishes linearly when $r \rightarrow r_0$, f_s^2 tends to zero logarithmically as $r_\Lambda \rightarrow r_0$. Therefore, we have to keep r_Λ slightly outside the horizon radius r_0 in our calculations. In other words, we have to take the low-energy (hydrodynamic) limit before the $r_\Lambda \rightarrow r_0$ limit. The precise value of f_s^2 , however, is not important for the final value of the diffusion constant.

III. HYDRODYNAMICS AND EMERGENT GRAVITY

We now generalize the discussion in Sec. II to the case of hydrodynamic modes in a finite-temperature plasma. Instead of the gauge field A_μ in the bulk, we now have the gravitational field. The emergent U(1) gauge field a_μ is now replaced by gravitational perturbations living on a surface near the horizon. Hydrodynamics, therefore, is a theory of a Goldstone boson, bifundamental with respect to two gravities. Such a Goldstone boson was considered in Ref. [26]. It is a map between the “boundary coordinates” x^μ and “horizon coordinates” X^M . Thus X^M can be thought of as 4 scalar fields living on the boundary coordinates x^μ ,

$$X^M = X^M(x^\mu), \quad (16)$$

and x^μ can be viewed as fields living on the horizon,

$$x^\mu = x^\mu(X^M). \quad (17)$$

We will use μ for the spacetime coordinates on the boundary, M for spacetime coordinates on the horizon, i, j for spatial coordinates on the boundary and a, b, \dots for the spatial coordinates on the horizon. We assume the boundary to be a four-dimensional spacetime, but the discussion can be generalized to any number of dimensions.

The ground state corresponds to $X^M = \delta_\mu^M x^\mu$, around which one can expand $X^M = \delta_\mu^M x^\mu + \phi^M(x^\mu)$. The fields ϕ^M then fluctuate around zero.

The Goldstone boson is coupled to the metric on the boundary $g_{\mu\nu}$. On the horizon, the metric is degenerate. The X^M space is what we will call a “Galilei space,” and is described in the Appendix A. Such a space is characterized by a degenerate metric $G_{MN}(X)$ and a null vector $n^M(X)$, so that $G_{MN}n^N = 0$. Alternatively, one can describe the Galilei space in terms of a spatial metric G_{ab} , a vector field v^a , and a Galilei clock factor γ , which are all functions of X^M :

$$ds^2 = G_{MN} dX^M dX^N = G_{ab} (dX^a - v^a dT) (dX^b - v^b dT), \quad n^N = \frac{1}{\gamma} (1, v^a). \quad (18)$$

A. Ideal hydrodynamics as a theory of Goldstone bosons

The action of the Goldstone boson should be invariant with respect to reparametrization of x^μ and of X^M . One could, in principle, derive this action from the gravity action in the bulk. We will, however, guess the form of this action by improving on the previous proposal of Ref. [27] (see also Ref. [28]). We first introduce the notion of \det_3 . Assume A is a 4×4 matrix, then

$$\det_3 A = \frac{1}{6}(\text{tr } A)^3 - \frac{1}{2} \text{tr } A \text{tr } A^2 + \frac{1}{3} \text{tr } A^3. \quad (19)$$

The operation \det_3 is defined so that if A is a matrix with one zero eigenvalue, then $\det_3 A$ is the product of three other eigenvalues. The action for the Goldstone boson is

$$S_0 = - \int d^4x \sqrt{-g} \epsilon \left(\sqrt{\det_3 (OG)} \right), \quad (20)$$

where $\epsilon(\dots)$ is a function of one variable, and the 4×4 matrix O is defined as

$$O^{MN} = g^{\mu\nu} \partial_\mu X^M \partial_\nu X^L, \quad (21)$$

and $(OG)^M_N \equiv O^{ML} G_{LN}$. Clearly, S_0 is invariant with respect to diffeomorphisms of x^μ and X^M spaces.

We can rewrite the action (20) in three-dimensional language by introducing

$$B^{ab} = g^{\mu\nu} e_\mu^a e_\nu^b, \quad e_\mu^a = \partial_\mu X^a - v^a \partial_\mu T. \quad (22)$$

A property of B^{ab} is that $\text{tr}_{4 \times 4} (OG)^n = \text{tr}_{3 \times 3} (BG)^n$, hence $\det_3 (OG) = \det (BG)$. Thus, the action can be written as

$$S_0 = - \int d^4x \sqrt{-g} \epsilon \left(\sqrt{\det B^{ab}} \sqrt{\det G_{ab}} \right). \quad (23)$$

If we set the metric to $G_{ab} = \delta_{ab}$, $v_a = 0$, then the action is the same as that of Ref. [27],

$$S_0 = - \int d^4x \sqrt{-g} \epsilon \left(\det^{1/2} [g^{\mu\nu} \partial_\mu X^a \partial_\nu X^b] \right). \quad (24)$$

Equation (20) generalizes the Lagrangian (24) to take into account the coupling with the emergent metric. In Appendix B we check, using holography, that the Lagrangian (20) correctly encodes the response of the fluid to homogeneous perturbations of the external metrics.

The conventional formulation of relativistic fluid dynamics is recovered in the unitary gauge $X^M = \delta_\mu^M x^\mu$. In this gauge, all the information about the fluid is contained in the horizon metric G_{MN} . It is clear from Eq. (23) that the action depends only on four parameters: three components of v^a and $\det G_{ab}$. In other words, S_0 has an additional gauge invariance with respect to arbitrary changes of G_{ab} that preserve the determinant. Using this extra invariance, we can fix the form of the horizon metric to

$$G_{MN} = s^{2/3} (\eta_{MN} + u_M u_N), \quad \eta_{MN} = \text{diag}(-1, 1, 1, 1), \quad (25)$$

where u^M satisfies $(u^0)^2 - (u^i)^2 = 1$. The argument of ϵ in Eq. (20) then becomes s . If we now identify parameter s with the local entropy density, u^M with the local fluid velocity, and $\epsilon(s)$ with the energy density (as a function of the entropy density), then the stress-energy tensor, computed by differentiating S_0 with respect to $g_{\mu\nu}$, has the same form as the stress-energy tensor of a ideal fluid,

$$T_0^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu}. \quad (26)$$

Here $P = s\epsilon'(s) - \epsilon$ is the pressure. For a conformal fluid, $\epsilon(s) \sim s^{4/3}$.

Note that Eq. (25) implies that the total entropy is equal to the volume of the horizon, without the factor of $1/4G$, but the factor can be reinserted without any problem. We can also take the alternative point of view that we set $4G = 1$ in all formulas.

B. Dissipation from coupling to Galilei metric

The action (23) does not contain the coupling to shear fluctuations of the horizon metric G_{ab} (those which preserve $\det G_{ab}$). Thus, S_0 is analogous to the time-derivative term $f_t^2(\partial_0\phi - A_0 + a_0)^2$ in the action (6) in Sec. II. We therefore expect that it is not the full Goldstone boson action S_{UV} —there must be another term, analogous to the spatial derivative term $f_s^2(\partial_i\phi - A_i + a_i)^2$ in Eq. (6). This term couples the Goldstone boson with the shear fluctuations of G_{ab} , and will be called $S_{\text{shear}}[X, g^{\mu\nu}, G_{MN}]$,

$$S_{UV} = S_0 + S_{\text{shear}}. \quad (27)$$

As in the gauge theory case where $f_s \rightarrow 0$ as $r_\Lambda \rightarrow r_0$, we expect S_{shear} to vanish in the limit $r_\Lambda \rightarrow r_0$, but this limit cannot be taken before the hydrodynamic limit.

We will assume the most general form for S_{shear} dictated by general coordinate invariance,

$$S_{\text{shear}} = \int d^4x \sqrt{-g} \mathcal{L}_1(\text{tr}(BG), \text{tr}(BG)^2, \text{tr}(BG)^3). \quad (28)$$

Here $\mathcal{L}_1(x_1, x_2, x_3)$ is an arbitrary function of $x_n = \text{tr}(BG)^n$. We will limit ourselves to conformal field theories, so \mathcal{L}_1 transforms like $\mathcal{L}_1 \rightarrow e^{4\omega} \mathcal{L}_1$ under Weyl transforms $g_{\mu\nu} \rightarrow e^{-2\omega} g_{\mu\nu}$. This means

$$\sum_n n x_n \frac{\partial \mathcal{L}_1}{\partial x_n} = 2\mathcal{L}_1. \quad (29)$$

Since one can add an arbitrary constant to \mathcal{L}_1 , without losing generality we can require that $\mathcal{L}_1 = 0$ when BG is proportional to the identity matrix, $BG = s^{2/3} \mathbb{1}$,

$$\mathcal{L}_1(x_n)|_{x_n=3s^{2n/3}} = 0. \quad (30)$$

Conformal invariance then implies

$$\sum_n n x_n \frac{\partial \mathcal{L}_1}{\partial x_n} \bigg|_{x_n=3s^{2n/3}} = 0. \quad (31)$$

We shall assume that the new term S_{shear} favors energetically configurations with equal eigenvalues of BG . Thus in equilibrium $BG = s^{2/3}\mathbb{1}$; in the hydrodynamic regime the deviation from equilibrium is small,

$$G = s^{2/3}B^{-1} + \delta G, \quad \delta G \ll s^{-2/3}B^{-1}. \quad (32)$$

Taking variations of \mathcal{L}_1 with respect to $g_{\mu\nu}$ and G_{ab} , we find its contribution to the boundary and horizon stress-energy tensors,

$$T_{\mu\nu}^{\text{shear}} = g_{\mu\alpha}g_{\nu\beta} \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}} = -2e_\mu^a e_\nu^b \sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} [G(BG)^{n-1}]_{ab} + g_{\mu\nu} \mathcal{L}_1, \quad (33)$$

where e_μ^a is defined in Eq. (22), and

$$\tau_{ab}^{\text{shear}} = G_{ac}G_{bd} \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{shear}}}{\delta G_{cd}} = 2 \sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} [G(BG)^n]_{ab}. \quad (34)$$

If we replace in Eqs. (33) and (34) $G \rightarrow G_0 \equiv s^{2/3}B^{-1}$, then both stress tensors vanish, due to Eq. (31). Therefore, both stress tensors are proportional to δG . To relate them to each other, we notice that, to leading order in δG ,

$$\sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} G(BG)^n = \sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} G(BG)^{n-1} (BG_0) + \sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} G_0 (BG_0)^{n-1} (B\delta G). \quad (35)$$

The second sum in the right hand side vanishes due to Eq. (31). We thus have

$$\sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} G(BG)^n = s^{2/3} \sum_n n \frac{\partial \mathcal{L}_1}{\partial x_n} G(BG)^{n-1}, \quad (36)$$

and the relationship between the boundary and the horizon stress tensors arising from S_{shear} is

$$T_{\mu\nu}^{\text{shear}} = -\left(e_\mu^a e_\nu^b - \frac{1}{4}g_{\mu\nu}O^{ab}\right) s^{-2/3} \tau_{ab}^{\text{shear}}. \quad (37)$$

Note that $T_{\mu\nu}^{\text{shear}}$ is traceless.

We now relate the stress tensor τ_{ab}^{shear} to the stress tensor of the IR theory of the horizon degrees of freedom. We make the shear fluctuations of G_{ab} dynamical, which means that the variation of the action with respect to these fluctuations vanishes,

$$\delta(S_{\text{shear}} + S_{\text{IR}}) = \int dx \sqrt{-g} \tau_{\text{shear}}^{ab} \delta G_{ab} + \int dT d^3X \gamma \sqrt{G} \hat{\tau}_{\text{hor}}^{ab} \delta G_{ab} = 0, \quad (38)$$

for all δG_{ab} which satisfies $G^{ab}\delta G_{ab} = 0$. Here $\hat{\tau}_{\text{hor}}^{ab}$ is the stress tensor of the degrees of freedom living on the horizon. This implies

$$\tau_{\text{shear}}^{ab} = -\frac{\det|\partial_\mu X^M| \gamma \sqrt{\det G_{ab}}}{\sqrt{-g}} (\hat{\tau}_{\text{hor}}^{ab} + \lambda' G^{ab}), \quad (39)$$

where λ' is undetermined. Moreover, the membrane paradigm [22–24] implies that the stress tensor at the horizon τ_{ab} is proportional to the projected tensor C_{ab} defined in the Appendix,

$$\hat{\tau}_{ab}^{\text{hor}} = -\eta_0 \left(C_{ab} - \frac{1}{3} G_{ab} C \right) - \zeta_0 G_{ab} C, \quad C_{ab} = \frac{1}{\gamma} (\nabla_a v_b + \nabla_b v_a + \dot{G}_{ab}), \quad C \equiv G^{ab} C_{ab}, \quad (40)$$

where η_0 and ζ_0 are the shear and bulk viscosities of the horizon. From the “membrane paradigm” we have $\eta_0 = 1/4\pi$ ($= 1/16\pi G$), the value of ζ_0 is not important since the coefficient λ in Eq. (39) is undetermined. Combining Eqs. (37), (39), and (40), we find the additional contribution to the stress-energy boundary tensor to be

$$T_{\mu\nu}^{\text{shear}} = -\frac{\eta_0}{s^{2/3}} \frac{\det |\partial_\mu X^M| \gamma \sqrt{\det G_{ab}}}{\sqrt{-g}} \left(e_\mu^a e_\nu^b - \frac{1}{4} g_{\mu\nu} O^{ab} \right) (C_{ab} + \lambda G_{ab}), \quad (41)$$

where λ is an undetermined coefficient. This equation can be rewritten in the 4-dimensional horizon form as

$$T_{\mu\nu}^{\text{shear}} = -\frac{\eta_0}{s^{2/3}} \frac{\det |\partial_\mu X^M| \gamma \sqrt{\det G_{ab}}}{\sqrt{-g}} \left(\partial_\mu X^M \partial_\nu X^N - \frac{1}{4} g_{\mu\nu} O^{MN} \right) (C_{MN} + \lambda G_{MN}). \quad (42)$$

Now going to the unitary gauge $X^M = x^\mu \delta_\mu^M$, substituting $G_{MN} = s^{2/3} (g_{MN} + u_M u_N)$, and choosing $n^M = u^M$ (using the fact that λC_{MN} is independent of the clock factor γ), we will find

$$\begin{aligned} T^{\mu\nu} = & -\eta P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \partial \cdot u \right) \\ & - \frac{\eta}{12} (g^{\mu\nu} + 4u^\mu u^\nu) P^{\alpha\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha + 2(u \cdot \partial) \ln s + \lambda P_{\alpha\beta}), \quad \eta = \eta_0 s, \end{aligned} \quad (43)$$

where $P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$. The second term in the Eq. (43), proportional to $g^{\mu\nu} + 4u^\mu u^\nu$, can be absorbed into the ideal part of the stress-energy tensor $(\epsilon + P)u^\mu u^\nu + P u^{\mu\nu} = (g^{\mu\nu} + 4u^\mu u^\nu)P$ by a redefinition of the temperature. We reproduce here the standard dissipative part of the stress-energy tensor, with the viscosity equal to $\eta_0 s$. Since on the horizon $\eta_0 = 1/4\pi$, this implies $\eta/s = 1/4\pi$.

IV. HOLOGRAPHIC ZERO SOUND

In this section, we apply the philosophy developed above to a zero-temperature case: the D3-D7 system at finite baryon density. The field theoretical description of such a system is the $\mathcal{N} = 4$ super Yang-Mills theory with $\mathcal{N} = 2$ fundamental matter. We assume the number of matter flavors N_f to be much smaller than the number of colors N_c , $N_f \ll N_c$, so that the probe approximation works on the gravity side of the duality.

The calculation of the current-current correlation function in this system reveals a zero-temperature mode, which was called the holographic zero-temperature sound, or, in analogy with a collective mode in the Fermi liquid, the zero sound [20]. This mode is different from zero-temperature collective excitations encountered in many-body theory. It has a linear

dispersion relation $\omega = vk$, with the velocity $v = (\partial P / \partial \epsilon)^{1/2}$ [20, 29]. Such a velocity would be natural if the mode was the superfluid Goldstone boson arising from the spontaneous breaking of the baryon number symmetry [30], but there is no indication that this breaking takes place in the geometry. Moreover, the damping rate of the mode is $\Gamma \sim k^2$, with a coefficient which is not suppressed by N_c , which is also inconsistent with the superfluid Goldstone boson interpretation.

In light of what we know by now, the nature of the holographic zero sound should be clear. This mode is the Goldstone boson, but not of the breaking of the global $U(1)$ baryon symmetry, but rather one of the breaking of a $U(1)_{\text{global}} \times U(1)_{\text{gauge}}$ symmetry down to a diagonal $U(1)$. The imaginary part in the dispersion curve of the Goldstone boson is due to the coupling of the dynamical $U(1)$ field to an infrared sector.

Our starting point will be the quadratic action for longitudinal gauge-field fluctuations in the bulk (cf. Ref. [20]):

$$S = \frac{\mathcal{N}_q}{2} \int d^{p+1}x \, dz \, z^{2-p} \left[f^3(z) (\partial_z a_0 - \partial_0 a_z)^2 + f(z) (\partial_0 a_i - \partial_i a_0)^2 - f(z) (\partial_z a_i - \partial_i a_z)^2 \right]. \quad (44)$$

Here p is the number of spatial dimensions of the field theory, and $f(z) = (1 + z^{2p})^{1/2}$ (which corresponds to a fixed charge density, equal to \mathcal{N}_q times a dimensionless constant). The UV corresponds to $z = 0$ and the IR to $z = \infty$. Here we assume the fundamental quarks to be massless.

We are interested in low-energy physics only. We will choose some value $z_\Lambda \gg 1$, so that $1/z_\Lambda$ is the cutoff of the low-energy effective theory. The degrees of freedom of the theory can be broken into the IR degrees of freedom, denoted collectively as ψ , living in $z > z_\Lambda$; the “emergent gauge field” living on the slice $z = z_\Lambda$, $a_\mu = a_\mu(z_\Lambda)$, and the UV degrees of freedom, collectively denoted as ϕ_{UV} living in $z < z_\Lambda$. The IR fields couple to a_μ , while the UV fields couple to both $A_\mu = a_\mu(0)$ and a_μ . The partition function of the theory, in external fields, can be written as:

$$Z[A_\mu] = \int D\psi \, Da_\mu \, D\phi_{\text{UV}} \exp(iS_{\text{IR}}[\psi, a_\mu] + iS_{\text{UV}}[\phi_{\text{UV}}, a_\mu, A_\mu]). \quad (45)$$

As discussed in Sec. II, due to the IR cutoff at z_Λ , S_{UV} describes an infinite tower of hadrons in a confining theory. In complete analogy with Sec. II, the only hadron relevant for the low-energy physics is the Goldstone boson of the spontaneous breaking $U(1) \times U(1) \rightarrow U(1)$. Thus, the effective action should now be

$$S = S_{\text{IR}}[\psi, a_\mu] + \int d^4x \, \frac{1}{2} \left[f_t^2 (\partial_0 \phi - A_0 + a_0)^2 - f_s^2 (\partial_i \phi - A_i + a_i)^2 \right]. \quad (46)$$

The decay constants f_t^2 and f_s^2 can be determined by the same method used to derive

Eqs. (15). We find,

$$f_t^2 = \mathcal{N}_q \left(\int_0^{z_\Lambda} \frac{dz}{z^{2-p} f^3(z)} \right)^{-1} = \mathcal{N}_q \frac{2\sqrt{\pi} p^2}{\Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2p}\right)}, \quad (47)$$

$$f_s^2 = \mathcal{N}_q \left(\int_0^{z_\Lambda} \frac{dz}{z^{2-p} f(z)} \right)^{-1} = \mathcal{N}_q \frac{2\sqrt{\pi} p}{\Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2p}\right)}. \quad (48)$$

(The integrals converge at large z , and the upper limit of integration $z_\Lambda \gg 1$ can be replaced by ∞ .)

Now we consider the IR sector. We will first give the final description of this sector, leaving the justification for later. The IR sector consist of an infinite set of (0+1)-d CFTs, one at each spatial point \mathbf{x} : $S_{\text{IR},\mathbf{x}}$. Each CFT therefore contains fields that depends only on time, which we denote collectively as $\psi_{\mathbf{x}}(t)$. Each CFT contains operators $O_{\mathbf{x},i}(t)$, $i = 1 \dots p$ with dimension 1, and also fields λ_i . The whole Lagrangian is

$$S = S_{\text{Goldstone}} + \mathcal{N}_q \int d\mathbf{x} \int dt \left\{ L_{(0+1)\text{dCFT}}[\psi_{\mathbf{x}}(t)] + O_{i,\mathbf{x}}(t) \dot{\lambda}_{i,\mathbf{x}}(t) + \lambda_{i,\mathbf{x}}(t) f_{0i}(t, \mathbf{x}) \right\}, \quad (49)$$

$$S_{\text{Goldstone}} = \int d^4x \frac{1}{2} [f_t^2 (\partial_0 \phi - A_0 + a_0)^2 - f_s^2 (\partial_i \phi - A_i + a_i)^2], \quad (50)$$

where we factor out \mathcal{N}_q from the Lagrangian of the (0+1)d CFT.

Integrating out the ψ degrees of freedom, one gets the following effective Lagrangian

$$S = S_{\text{Goldstone}} + \mathcal{N}_q \int d\mathbf{x} \left[\frac{i}{2} \int \frac{d\omega}{2\pi} |\omega|^3 |\lambda_{i,\mathbf{x}}(\omega)|^2 + \int dt \lambda_{i,\mathbf{x}}(t) f_{0i}(t, \mathbf{x}) \right], \quad (51)$$

and integrating over λ_i , one gets

$$S = S_{\text{Goldstone}} + \frac{\mathcal{N}_q}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{|\omega|^3} |f_{0i}(\omega, \mathbf{q})|^2. \quad (52)$$

Now choosing, e.g., the $a_0 = 0$ gauge, and diagonalizing a 2×2 matrix for ϕ and a_i , one gets the dispersion relation for the zero sound,

$$\omega = vq - i\gamma q^2, \quad \gamma = \frac{f_s^2 v^2}{2\mathcal{N}_q}, \quad (53)$$

where $v = f_s/f_t = 1/\sqrt{p}$, and the coefficient γ in the imaginary part is

$$\gamma = \frac{f_s^2 v^2}{2\mathcal{N}_q} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2p}\right)}. \quad (54)$$

The result for the dispersion relation coincides with that of Ref. [20], which is a check of the validity of the effective theory (49).

Now let us justify Eq. (49). In the $a_z = 0$ gauge, the field equations in the full metric are

$$(f^3 z^{2-p} a'_0)' - f z^{2-p} (q^2 a_0 + \omega q_i a_i) = 0, \quad (55)$$

$$(f z^{2-p} a'_i)' + f z^{2-p} (\omega q_i a_0 + \omega^2 a_i) = 0. \quad (56)$$

In the regime $z \gg 1$, $f = z^p$. One can see that a_0 changes with z so slowly that it can be considered a constant. Then $\tilde{a}_i = a_i + q_i a_0 / \omega$ satisfied the equation

$$(z^2 \tilde{a}'_i)' + z^2 \omega^2 \tilde{a}_i = 0. \quad (57)$$

Changing variable to $\phi_i = z \tilde{a}_i$, we see that ϕ satisfies the equation of a massless scalar in AdS_2 . The action for ϕ_i is

$$S = \frac{\mathcal{N}_q}{2} \int d^{p+1} x dz [(\partial_0 \phi_i)^2 - (\partial_z \phi_i)^2]. \quad (58)$$

There are two CFTs corresponding to (58) [31]. In the first CFT the operator O dual to ϕ has dimension 1 and correlation function (in Euclidean space) $\langle OO \rangle = \mathcal{N}_q |\omega|$; in the second CFT, O has dimension 0 and $\langle OO \rangle = \mathcal{N}_q |\omega|^{-1}$. (The coupling of O and ϕ is taken to be $\mathcal{N}_q \phi O$, so that \mathcal{N}_q factors out of the action.)

To determine the dimension of the operator dual to ϕ , let us first assume $a_0 = 0$, for simplicity, so $\tilde{a}_i = a_i$. The boundary condition for ϕ , for $1 \ll z \ll 1/\omega$, is $\phi = a_i z + \dots$, which is the more regular asymptotics near the boundary (the other one is z^0). Therefore the emergent electric gauge field a_i serves as the source for the operator dual to ϕ , and the dimension of that operator is 0. Hence our model is

$$S = \mathcal{N}_q \left[S_{(0+1)\text{dCFT}} - \int d\mathbf{x} \int dt a_i(t, \mathbf{x}) O_{i,\mathbf{x}}^{\Delta=0}(t) \right] + S_{\text{Goldstone}}. \quad (59)$$

This is the action written in the $a_0 = 0$ gauge. To restore gauge invariance, we can introduce a Legendre multiplier to enforce the constraint $\partial_t \tilde{a}_i = f_{0i}$:

$$S = \mathcal{N}_q \left[S_{(0+1)\text{dCFT}} - \int d\mathbf{x} \int dt \tilde{a}_i(t, \mathbf{x}) O_{i,\mathbf{x}}^{\Delta=0}(t) - \int d^4 x \lambda_{i,\mathbf{x}} (\partial_t \tilde{a}_i(x) - f_{0i}(x)) \right] + S_{\text{Goldstone}}. \quad (60)$$

Now we note that to integrate over \tilde{a}_i is to take a Legendre transform and convert $(0+1)d$ CFT into a CFT with scalar operator of dimension 1. In this way we arrive to Eq. (49).

V. CONCLUSION

In this paper, we have been advocating the point of view that holographic liquids can be described, at long distances, by a theory of Goldstone bosons coupled to an infrared sector through emergent gauge and gravitational fields. We consider in this paper only a few simplest examples. It should be possible to extend the calculation in this paper to other cases, for example for the R-charged black holes, where the relationship between boundary

and horizon kinetic coefficients is not trivial [32]. Possibly, the most interesting applications of our formalism are zero-temperature systems: the holographic superfluids and the system dual to the extremal Reissner-Nordström black hole. The latter plays a central role in recent construction of holographic non-Fermi liquids. In the case of the extremal Reissner-Nordström black holes, it has been found that the Kubo's formulas yield finite values for the kinetic coefficients (for example, the shear viscosity η) [33]. However, the effective low-energy description of extremal Reissner-Nordström black holes cannot be hydrodynamics. In hydrodynamics, there is a formula for entropy production (in local fluid rest frame)

$$\partial_\mu s^\mu = \frac{\eta}{T} \left(\partial_i u_j + \partial_j u_i - \frac{2}{3} \partial \cdot u \right)^2. \quad (61)$$

This formula does not make sense if η is finite in the limit $T \rightarrow 0$: the rate of entropy production would be infinite. The effective field theory therefore has to be of a different nature. It seems that the effective theory has to involve Goldstone modes, coupled with AdS_2 degrees of freedom. However, the details of this theory need to be worked out.

The new point of view on holographic liquids reduces the problem of finding the low-energy dynamics of such liquids into finding the Goldstone boson degrees of freedom, the horizon degrees of freedom, and the manner they are coupled together. The appearance of the emergent gauge fields brings an interesting questions about the possible relationships between recent constructions of holographic liquids with the older attempts to construct nontrivial low-energy effective theories of strongly correlated electrons or spin systems, which typically involve a “deconfinement” of emergent gauge fields. Hopefully, our work will help bridging the gap between holographic models and the field-theoretical models for strongly coupled electronic systems.

The authors thank A. O'Bannon, A. Karch, Hong Liu, J. Polchinski, and A. Strominger for discussions. This work is supported, in part, by DOE grant DE-FG02-00ER41132.

Appendix A: Galilei spacetime and Galilei field theories

1. Galilei spacetime

By “Galilei spacetime” we have in mind a structure consisting of manifold with a degenerate metric

$$ds^2 = G_{MN} dx^M dx^N, \quad G_{MN} n^M = 0, \quad (A1)$$

and a Galilean clock factor $\gamma(T, X)$. We will say that the combination (G_{MN}, γ) defines a Galilei spacetime. The null metric can be parameterized by the null vector v^a and a spatial metric G_{ab} ,

$$ds^2 = G_{ab} (dX^a - v^a dT) (dX^b - v^b dT), \quad (A2)$$

and so the Galilei space can be said to be characterized by (G_{ab}, v^a, γ) .

The Galilei spacetime can be considered as a limit $\epsilon \rightarrow 0$ of a spacetime with a metric

$$ds_\epsilon^2 = G_{MN}^\epsilon dX^M dX^N = -\epsilon^2 \gamma^2 dT^2 + G_{ab}(dX^a - v^a dT)(dX^b - v^b dT). \quad (\text{A3})$$

All quantities for the Galilei structure should be defined to be finite in the limit $\epsilon \rightarrow 0$. For example, the volume element is defined as

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \sqrt{-\det G_{MN}^\epsilon} d^4 X = \gamma \sqrt{G} dT d^3 X, \quad (\text{A4})$$

where $G \equiv \det G_{ab}$. The general coordinate transformations (diffeomorphisms) of the Galilei spacetime can be obtained as the $\epsilon \rightarrow 0$ limit of the diffeomorphisms on the space (A3). One can easily work out the action of infinitesimal diffeomorphisms on the metric components. Under spatial transforms, $X^a \rightarrow X'^a = X^a + \xi^a$,

$$\delta G_{ab} = -\xi^c \partial_c G_{ab} - G_{cb} \partial_a \xi^c - G_{ac} \partial_b \xi^c, \quad (\text{A5})$$

$$\delta v^a = -\xi^c \partial_c v^a + v^c \partial_c \xi^a + \dot{\xi}^a, \quad (\text{A6})$$

$$\delta \gamma = -\xi^c \partial_c \gamma, \quad (\text{A7})$$

and under time transforms, $T \rightarrow T' = T + \xi$,

$$\delta G_{ab} = -\xi \dot{G}_{ab} + v_a \partial_b \xi + v_b \partial_a \xi, \quad (\text{A8})$$

$$\delta v^a = -\xi \dot{v}^a - v^a d_T \xi, \quad (\text{A9})$$

$$\delta \gamma = -\xi \dot{\gamma} - \gamma d_T \xi, \quad (\text{A10})$$

where $d_T \xi \equiv \dot{\xi} + v^c \partial_c \xi$. These can be taken as the intrinsic definition of diffeomorphisms of the Galilei space, without referring to the limiting procedure $\epsilon \rightarrow 0$. The action of the Goldstone boson should be invariant with respect to diffeomorphisms of both the physical spacetime x^μ and the Galilei spacetime X^M .

Under diffeomorphisms, contravariant vectors and tensors transform as

$$\delta A^M = -\xi^L \partial_L A^M + A^L \partial_L \xi^M, \quad (\text{A11})$$

$$\delta A^{MN} = -\xi^L \partial_L A^{MN} + A^{LN} \partial_L \xi^M + A^{ML} \partial_L \xi^N, \quad (\text{A12})$$

while covariant vectors and tensors transform as

$$\delta A_M = -\xi^L \partial_L A_M - A_L \partial_M \xi^L, \quad (\text{A13})$$

$$\delta A_{MN} = -\xi^L \partial_L A_{MN} - A_{LN} \partial_M \xi^L - A_{ML} \partial_N \xi^L. \quad (\text{A14})$$

The Galilei space possesses one intrinsic vector field

$$n^M = \frac{1}{\gamma} (1, v^a). \quad (\text{A15})$$

One can check that n^M transforms like a vector under diffeomorphisms. It is a null vector: $G_{MN} n^M = 0$. In fact one can take the pair (G_{MN}, n^M) as the definition of the Galilei space.

Since the Galilei metric is degenerate, indices can be lowered using G_{MN} but, in general, cannot be raised. Tensors obtained by lowering the indices of a fully contravariant tensor are perpendicular to the null vector,

$$A_{MN} = G_{MK}G_{NL}A^{KL} \Rightarrow n^M A_{MN} = n^N A_{MN} = 0. \quad (\text{A16})$$

Such a tensor is completely determined by its spatial components: $A_{0a} = -A_{ab}v^b$, $A_{00} = A_{ab}v^a v^b$. One can regard the spatial three-tensor A_{ab} as an object by itself, which we will call a projected tensor. Under spatial reparametrization it transforms as a conventional tensor in three-dimensional space, and under time reparametrization it transforms as

$$T \rightarrow T + \xi : \quad \delta \hat{A}_{ab} = -\xi \partial_t \hat{A}_{ab} + \partial_a \xi v^c \hat{A}_{cb} + \partial_b \xi v^c \hat{A}_{ac}. \quad (\text{A17})$$

The metric tensor G_{ab} is one such tensor. The indices of a projected tensor can be raised by using the inverse spatial metric G^{ab} : $\hat{A}^{ab} = G^{ac}G^{bd}A_{cd}$. This fully contravariant projected tensor transforms under time reparametrization as

$$T \rightarrow T + \xi : \quad \delta \hat{A}^{ab} = -\xi \partial_t \hat{A}^{ab} - v^a \partial_c \xi \hat{A}^{cb} - v^b \partial_c \xi \hat{A}^{ac}. \quad (\text{A18})$$

G^{ab} is a contravariant projected tensor. Note that a contravariant projected tensor does not corresponds uniquely to a four-tensor, rather, it corresponds to a whole class of four-tensors which differ from each other by $A^{MN} \rightarrow A^{MN} + n^M k^N + n^N k^M$.

We can construct, in analogy with the extrinsic curvature, the following symmetric tensor,

$$C_{MN} = 2\nabla_{(M}n_{N)} = 2G_{L(M}\partial_{N)}n^L + n^L\partial_L G_{MN}. \quad (\text{A19})$$

Since $n^M C_{MN} = 0$, C_{ab} is a projected tensor. In components,

$$C_{ab} = \frac{1}{\gamma}(2\nabla_{(a}v_{b)} + \dot{G}_{ab}). \quad (\text{A20})$$

This tensor is proportional to the inverse of the Galilei clock factor γ .

2. Stress-energy tensor in Galilei field theories

For a quantum field theory in Galilei space, one can define the stress-energy tensor by taking small variation of the action with respect to the external metric,

$$\delta S = \frac{1}{2} \int dT d^3X \gamma \sqrt{\det G_{ab}} \tau^{MN} \delta G_{MN}. \quad (\text{A21})$$

Since the matrix G_{MN} is constrained to be degenerate, the stress-energy tensor τ^{MN} is defined up to one arbitrary contribution,

$$\tau^{MN} \rightarrow \tau^{MN} + \lambda n^M n^N, \quad (\text{A22})$$

so there are 9 independent components of τ^{MN} in four dimensions. By lowering the indices $\tau_{MN} = G_{MA}G_{NB}\tau^{AB}$, one obtains a transverse tensor τ_{ab} . Note that τ_{ab} contains less information than τ^{MN} : there are three extra independent components in τ^{MN} . This can be seen by rewriting Eq. (A21) in components,

$$\delta S = \int dT d^3X \gamma \sqrt{\det G_{ab}} \left(\frac{1}{2} \hat{\tau}^{ab} \delta G_{ab} + \rho_a \delta v^a \right), \quad (\text{A23})$$

where

$$\hat{\tau}^{ab} = \tau^{ab} - v^a \tau^{0b} - v^b \tau^{a0} + v^a v^b \tau^{00}, \quad (\text{A24})$$

$$\rho_a = v_a \tau^{00} - G_{ab} \tau^{0b}. \quad (\text{A25})$$

Appendix B: Matching effective theory with holography

In order to compare and match our discussion to an actual AdS/CFT calculation, we first work out the on-shell action of the holographic setup to quadratic order in metric fluctuations with boundary conditions imposed at the boundary as well as an intermediate cutoff scale. Following Refs. [34, 35] and its conventions, the thermal AdS background is given by

$$ds^2 = \frac{(\pi T R)^2}{u} (-f(u) dt^2 + d\vec{x}^2) + \frac{R^2}{4u^2 f(u)} du^2 = g_{MN}^{(0)} dX^M dX^N, \quad (\text{B1})$$

where $f(u) = 1 - u^2$. The boundary is at $u = 0$ and the horizon at $u = 1$. We denote by u_Λ the position of the stretched horizon, which separates the UV and IR parts of the metric. For the fluctuations, defined through $g_{MN} = g_{MN}^{(0)} + h_{MN}$, we introduce the parameterization

$$H_{tt} = \frac{u h_{tt}}{f(\pi T R)^2}, \quad H'_{uu} = \frac{u \sqrt{f} h_{uu}}{R^2}, \quad (H_{ij}, H_{ti}, H_{u\mu}) = \frac{u}{(\pi T R)^2} (h_{ij}, h_{ti}, h_{u\mu}). \quad (\text{B2})$$

In the gauge $H_{uM} = 0$ the boundary values of $H_{\mu\nu}$ in the on-shell action are then the sources of the dual stress-energy tensor.

As an aside and simple observation: The length $s = \int_{\tau_0}^{\tau_1} d\tau \sqrt{g_{MN} \partial_\tau X^M \partial_\tau X^N}$ of a trajectory $X^\mu(\tau) = u(\tau) \delta_u^\mu$ is shifted to linear order by $\delta s = L(H_{uu}(u(\tau_1)) - H_{uu}(u(\tau_0)))$.

We want to turn on constant external metric perturbations, keeping the Goldstone fields frozen at the vacuum value in Eq. (20). This is equivalent to fixing the boundary conditions $H_{\mu\nu}(0) = h_{\mu\nu}$, $H_{\mu\nu}(u_\Lambda) = 0$ in the gauge $H_{u\mu} = 0$ ($\mu, \nu \neq u$). The component H_{uu} requires a special treatment (see below).

Static, spatially homogeneous fluctuations decouple according to their respective spin. Spin-one fluctuations spanned by H_{ti} , H_{ui} and spin-two fluctuations spanned by the components of $\tilde{H}_{ij} = H_{ij} - \frac{1}{3} \delta_{ij} H_{kk}$ satisfy the linearized equations of motions

$$\begin{aligned} 0 &= H''_{ti}(u) - \frac{1}{u} H'_{ti}(u), \\ 0 &= \tilde{H}''_{ij}(u) - \frac{1+u^2}{u f(u)} \tilde{H}'_{zx}(u). \end{aligned} \quad (\text{B3})$$

The components H_{ui} drop out to linear order and can be consistently set to zero. With the boundary conditions $H(0) = h$, $H(u_\Lambda) = 0$, the equations are solved by

$$\begin{aligned} H_{ti}(u) &= h_{ti} \left(1 - \frac{u^2}{u_\Lambda^2} \right), \\ \tilde{H}_{ij}(u) &= \tilde{h}_{ij} \left(1 - \frac{\ln f(u)}{\ln f(u_\Lambda)} \right). \end{aligned} \quad (\text{B4})$$

Since static and homogeneous gauge transformations in these channels are generated by Killing vectors of the background, gauge transformations do not impose additional constraints on these solutions.

For the spin zero fluctuations, spanned by H_{tt} , H_{ii} , H_{ut} and H'_{uu} , the linearized Einstein equations yield

$$\begin{aligned} 0 &= Z''(u) - \frac{1}{u} Z'(u), \\ 0 &= (2 + f(u)) H'_{ii}(u) - 3f(u) H'_{tt}(u) + 24\sqrt{f(u)} H'_{uu}(u), \end{aligned} \quad (\text{B5})$$

where we introduced the gauge-invariant combination $Z(u) = (1 + u^2)H_{ii}(u) + 3f(u)H_{tt}(u)$. Similar as in the spin one case, H_{ut} drops out to linear order and can be set to zero. However, since one equation in (B5) is first order, we cannot set $H_{uu}(u) = 0$. For this reason we keep $H'_{uu}(u)$ arbitrary for the moment and recall that each choice of it defines a separate gauge. The solution for the equation for Z is

$$Z(u) = Z(0) \left(1 - \frac{u^2}{u_\Lambda^2} \right), \quad (\text{B6})$$

and hence we have

$$\begin{aligned} 0 &= (2 + f(u)) H'_{ii}(u) - 3f(u) H'_{tt}(u) + 24\sqrt{f(u)} H'_{uu}(u), \\ 0 &= (1 + u^2) H_{ii}(u) + 3f(u) H_{tt}(u) - Z(0) \left(1 - \frac{u^2}{u_\Lambda^2} \right), \end{aligned} \quad (\text{B7})$$

which yield

$$\begin{aligned} H_{tt}(u) &= \frac{(1 + u_\Lambda^2)}{6u_\Lambda^2} Z(0) + \frac{2(1 + u^2)H_{uu}(u)}{\sqrt{f(u)}}, \\ H_{ii}(u) &= \frac{(-1 + u_\Lambda^2)}{2u_\Lambda^2} Z(0) - 6\sqrt{f(u)} H_{uu}(u). \end{aligned} \quad (\text{B8})$$

To satisfy the boundary condition $H_{tt}(u_\Lambda) = H_{ii}(u_\Lambda) = 0$, we require

$$H_{uu}(u_\Lambda) = -\frac{\sqrt{f(u_\Lambda)}}{12u_\Lambda^2} Z(0). \quad (\text{B9})$$

Then the boundary conditions $H_{tt}(0) = h_{tt}$ and $H_{ii}(0) = h_{ii}$ can be achieved by choosing appropriate $Z(0)$ and $H_{uu}(0)$. It is worth noting that we can choose the metric perturbation

and derivatives in u to vanish at both boundaries and that $H_{uu}(0) - H_{uu}(u_\Lambda)$ shows up in the length of the trajectory mentioned above.

The gravity action is given by the sum

$$S = S_{\text{EH}} + S_{\text{GH}} + S_{\text{CT}}, \quad (\text{B10})$$

where the Einstein-Hilbert term S_{EH} , the Gibbons-Hawking term S_{GH} and the counter-term S_{CT} are defined as

$$\begin{aligned} S_{\text{EH}} &= \frac{N^2}{8\pi^2 R^3} \int_{u_\Lambda}^{u_\epsilon} du d^4x \sqrt{-g} \left(\mathcal{R} + \frac{12}{R^2} \right), \\ S_{\text{GH}} &= \frac{N^2}{4\pi^2 R^3} \int d^4x \sqrt{-\gamma} K|_{u_\Lambda}^{u_\epsilon}, \\ S_{\text{CT}} &= -\frac{3N^2}{4\pi^2 R^4} \int d^4x \sqrt{-\gamma}|^{u_\epsilon}. \end{aligned} \quad (\text{B11})$$

Here we introduced u_ϵ as a regulator of the renormalization scheme in order to have finite intermediate results and will take $u_\epsilon \rightarrow 0$ at the end of the calculation. Also note that the counterterm only contributes for $u = u_\epsilon$. Since the Einstein-Hilbert action also decomposes into surface terms for fluctuations obeying the equations of motions, i.e. $S_{\text{EH}} = S_{\text{EH},u=u_\epsilon}^{\text{boundary}} + S_{\text{EH},u=u_\Lambda}^{\text{boundary}}$, we have two contributions to the on-shell action:

$$S = \underbrace{(S_{\text{EH},u=u_\Lambda}^{\text{boundary}} + S_{\text{GH},u=u_\Lambda})}_{\equiv S_\Lambda} + \underbrace{(S_{\text{EH},u=u_\epsilon}^{\text{boundary}} + S_{\text{GH},u=u_\epsilon} + S_{\text{CT},u=u_\epsilon})}_{\equiv S_\epsilon}. \quad (\text{B12})$$

The evaluation of the on-shell action is tedious and we only quote the results. Requiring $H'_{uu}(0) = H''_{uu}(0) = 0$ we find for the contribution from the boundary in the limit $u_\epsilon \rightarrow 0$

$$\begin{aligned} S_\epsilon &= \frac{\pi^2 N^2 T^4 V}{8} \left[-1 + \frac{1}{2}(h_{ii} + 3h_{tt}) + \frac{1}{24}(h_{ii}^2 - 36h_{ti}^2 - 6\tilde{h}_{ij}\tilde{h}_{ij} + 6h_{ii}h_{tt} + 9h_{tt}^2 \right. \\ &\quad \left. - 8h_{ii}H''_{ii}(0) + 12h_{tt}H''_{ii}(0) - 24h_{ti}H''_{ti}(0) + 12h_{ij}\tilde{H}''_{ij}(0) + 12h_{ii}H''_{tt}(0) \right]. \end{aligned} \quad (\text{B13})$$

Here we already used the requirement that $H_{\mu\nu}(u)$ is finite at the boundary and even in u . The contribution S_Λ in the limit $u_\Lambda \rightarrow 1$ with $H'_{uu}(u_\Lambda) = H''_{uu}(u_\Lambda) = 0$ and $H_{\mu\nu} = 0$ vanish: $S_\Lambda = 0$. Therefore, after plugging in the equations of motions we obtain

$$S = \frac{\pi^2 N^2 T^4 V}{8} \left[-1 + \frac{1}{2}(h_{ii} + 3h_{tt}) + \frac{1}{24}(-3h_{ii}^2 + 12h_{ti}^2 - 6\tilde{h}_{ij}^2 - 6h_{ii}h_{tt} + 9h_{tt}^2) \right]. \quad (\text{B14})$$

This expression should be compared to the expansion of Eq. (20) in unitary gauge using $\epsilon(s) = Cs^{4/3}$. For $X^M = \delta_\mu^M x^\mu$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $G_{MN} = s^{2/3}(\eta_{MN} + \delta_M^0 \delta_N^0)$ we find, to quadratic order

$$S_{\text{fluid}} = \frac{Cs^{4/3}}{3} \left[-3 + \frac{1}{2}(h_{ii} + 3h_{tt}) + \frac{1}{24}(-3h_{ii}^2 + 12h_{ti}^2 - 6\tilde{h}_{ij}^2 - 6h_{ii}h_{tt} + 9h_{tt}^2) \right]. \quad (\text{B15})$$

We see a complete agreement between the expressions obtained from holography and effective field theory.

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